International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 6, Issue 4, Jun – Jul 2017; 83-98

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

PROPERTIES OF EXTENDED MATRIX ALGEBRA

JAYANTA BISWAS
Department of Mathematics, Barasat Government College, Kolkata, West Bengal, India

ABSTRACT

We have lots of properties of traditional matrix algebra. Some of these properties are studied and extended in [1-5]. Here we extend some more properties in extended matrix algebra on M (F), the set of all matrices over a given field F [1, 3]. Also following [1-11], we are motivated to introduce some new properties in extended matrix algebra on M (F).

KEYWORDS: Eigen Value, Extended Matrix Algebra

Notations

(i) $M_{m\times n}(F)$ denotes the set of all $m\times n$ matrices over a given field F.

(ii) $A_{m\times n}\in M(F)$ denotes $A_{m\times n}$ Is an $m\times n$ matrix in (F).

(iii) $M_n(F)$ denotes the set of all $n\times n$ matrices over a given field F.

(iv) $O_{m\times n}$ denotes the $m\times n$ matrix in $M(F)$, of which all the entries are zero.

(v) If $A_{m\times n}=\left(a_{ij}\right)_{m\times n}\in M(F)$ and p, q are positive integers such that $\leq m, q\leq n$, then $A_{p\times q}=\left(a_{ij}\right)_{p\times q}$.

1. INTRODUCTION

Matrix algebra and Linear spaces of linear transformations of vector spaces (considering linear transformations as matrices) are extended in [1, 3] as:

Definition (1.1) [1, 3]

Define 'addition' of matrices in $M(F)$ by for all $=\left(a_{ij}\right)_{m\times n}, B=\left(b_{ij}\right)_{p\times q}\in M(F)$, $A+B=\left(c_{ij}\right)_{r\times s}$, where $r=\max\{m,p\}$, $s=\max\ \{n,q\}$ and for $i=1,2,\dots.,r;\ j=1,2,\dots.s, c_{ij}=a'_{ij}+b'_{ij}$,

where $a'_{ij}=\begin{cases}a_{ij}\,, if\ 1\leq i\leq m\,, 1\leq j\leq n\\ 0\,, otherwise\end{cases}$, for $i=1,2,\dots.,r;\ j=1,2,\dots.,s$

and $b'_{ij}=\begin{cases}b_{ij}\,, if\ 1\leq i\leq p\,, 1\leq j\leq q\\ 0\,, otherwise\end{cases}$, for $i=1,2,\dots.,r$; $j=1,2,\dots.,s$.

Definition (1.2) [1, 3]

Define 'multiplication' of matrices in $M(F)$ by for all $A=\left(a_{ij}\right)_{m\times n}, B=\left(b_{ij}\right)_{p\times q}\in M(F)$, $AB=\left(c_{ij}\right)_{m\times q}$, where for $i=1,2,\dots.,m;\ j=1,2,\dots.,, c_{ij}=\sum_{k=1}^{\min\ \{n,p\}}a_{ik}b_{kj}$.

Definition (1.3) [1, 3]

For $m, n \in \mathbb{N}$, we define $I_{m\times n}$ as $I_{m\times n} = \left(\delta_{ij}\right)_{m\times n}$, where for $i = 1,2, \dots, m$; $j = 1, 2, \dots., n$, $\delta_{ij} = \begin{cases} 1, if\ i = j \\ 0, if\ i \neq j \end{cases}$.

Example (1.1): Let $A = \begin{pmatrix} 2 & 1 & 0 & -1 & 4 \\ 0 & 5 & 2 & 3 & 0 \\ -1 & 3 & -5 & 7 & 2 \end{pmatrix}$ and $= \begin{pmatrix} 1 & 3 \\ 0 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$.

Then $A + B = \begin{pmatrix} 3 & 4 & 0 & -1 & 4 \\ 0 & 6 & 2 & 3 & 0 \\ -2 & 5 & -5 & 7 & 2 \\ 2 & 1 & 0 & 0 & 0 \end{pmatrix}$ and $AB = \begin{pmatrix} 0 & 6 \\ 4 & 12 \\ 18 & -3 \end{pmatrix}$.

Definition (1.4) [1, 3]

Let R be a non-empty set on which two binary operations "addition and "multiplication are defined. Then the algebraic structure (R, +,.) is said to be a weak hemi-ring if

(i) (R, +) is a commutative monoid, the identity element is called 'zero', denoted by 0. (ii) (R,.) is a semi-group.

(iii) Multiplication is distributive over addition. (iv) $a.0 \neq 0, 0.a \neq 0$, in general, for $a \in R$.

Theorem (1.1) [1, 3]

The algebraic structure (M(F), +, .) is a weak hemi-ring having zero $O_{1\times1} = (0)_{1\times1}$.

Note (1.1) [3]

Let $m, n \in \mathbb{N}$ and $A = \left(a_{ij}\right)_{m\times n} \in M(F)$ be arbitrary. Let $I_{B(p,n)} = \begin{pmatrix} I_n \\ B_{p\times n} \end{pmatrix}$, where $B_{p\times n} \in M(F)$ is arbitrary and $p \in \mathbb{N}$ is arbitrary and let $I^{B(m,q)} = \left(I_m\ B_{m\times q}\right)$, where $B_{m\times q} \in M(F)$ is arbitrary and $q \in \mathbb{N}$ is arbitrary. Then it is clear that $A.I_{B(p,n)} = A$ and $I^{B(m,q)}.A = A$; but $I_{B(p,n)}.A \neq A$ & $A.I^{B(m,q)} \neq A$, in general. Again, $I_m.A = A = A.I_n$, But $I_n.A \neq A$ & $A.I_m \neq A$, if $m \neq n$.

Again, it is obvious that $I_{m\times n}A_{m\times p} = A_{m\times p}\ iff\ n \geq m\ and\ A_{p\times n}I_{m\times n} = A_{p\times n}\ iff\ m \geq n$; but

$I_{m\times n}A_{m\times p} = \begin{pmatrix} A_{n\times p} \\ O_{(m-n)\times p} \end{pmatrix} \neq A_{m\times p}, in\ general, and\ I_n A_{m\times p} = A_{n\times p} \neq A_{m\times p}\ if\ n < m$

Also, $A_{p\times n}I_{m\times n} = (A_{p\times m} \quad O_{p\times(n-m)}) \neq A_{p\times n}$, $in\ general, and\ A_{p\times n}I_m = A_{p\times m} \neq A_{p\times n}\ if\ m < n$

Now it can be easily proved that for given positive integers *m, n*, for all $A_{m\times n} \in M(F)$,

$A_{m\times n}I_{m\times n} = I_{m\times n}A_{m\times n} = A_{m\times n}$ iff $m = n$.

Theorem (1.2) [3]

$(M_{m\times n}, +, .)$ forms a ring with zero $O_{m\times n}$..

Definition (1.5) [3]

For a given nonzero matrix $A_{m\times n} \in M(F)$, if there exists a matrix $B_{p\times q} \in M(F)$ such that $A_{m\times n}B_{p\times q} = I_{m\times q}$ then $B_{p\times q}$ is called a right inverse of $A_{m\times n}$ and if $B_{p\times q}A_{m\times n} = I_{p\times n}$, then $B_{p\times q}$ is called a left inverse of $A_{m\times n}$.

Theorem (1.3) [3]

Let $m \leq n$. Then for two non-zero matrices $A_{m\times n}, B_{m\times n} \in M(F)$, $A_{m\times n}B_{m\times n} = I_{m\times n}$ iff $A_{m\times m}B_{m\times m} = B_{m\times m}A_{m\times m} = I_m$ and for $j = m+1, m+2, \dots, n$, each j^{th} column B_j (say) of $B_{m\times n}$ is zero.

Corollary (1.3.a): Let $n \leq m$. Then for two non-zero matrices $A_{m\times n}, B_{m\times n} \in M(F)$, $A_{m\times n}B_{m\times n} = I_{m\times n}$ iff $A_{n\times n}B_{n\times n} = B_{n\times n}A_{n\times n} = I_n$ and for $i = n+1, n+2, \dots, m$, each i^{th} row A_i (say) of $A_{m\times n}$ is zero.

Corollary (1.3.b): Let $m \leq n$. Then for two non-zero matrices $A_{m\times n}, B_{m\times n} \in M(F)$,

$A_{m\times n}B_{m\times n} = B_{m\times n}A_{m\times n} = I_{m\times n}$ iff $A_{m\times m}B_{m\times m} = B_{m\times m}A_{m\times m} = I_m$ and

for $j = m+1, m+2, \dots, n$, each j^{th} column of $A_{m\times n}$ and $B_{m\times n}$ are zero.

Corollary (1.3.c): Let $n \leq m$. Then for two non-zero matrices $A_{m\times n}, B_{m\times n} \in M(F)$, $A_{m\times n}B_{m\times n} = B_{m\times n}A_{m\times n} = I_{m\times n}$ iff $A_{n\times n}B_{n\times n} = B_{n\times n}A_{n\times n} = I_n$ and for $i = n+1, n+2, \dots, m$, each i^{th} row of $A_{m\times n}$ and $B_{m\times n}$ are zero.

Theorem (1.4) [3]

Let $A_{m\times n}, B_{p\times q} \in M(F)$. Then $A_{m\times n}B_{p\times q} = I_{m\times q}$ iff one of the following four conditions hold.

(i). $A_{m\times n}B_{n\times m} = I_m$ and $A_{m\times n}\left(B_{m+1}, B_{m+2}, \dots, B_q\right)_{p\times(q-m)} = O_{m\times(q-m)}$, if $n \leq p, m < q$; where for $j = m+1, m+2, \dots, q$; B_j is the j^{th} column of $B_{p\times q}$.

(ii). $A_{q\times n}B_{n\times q} = I_q$ and $\begin{pmatrix} R_{q+1} \\ R_{q+2} \\ \dots\dots \\ \dots\dots \\ R_m \end{pmatrix}_{(m-q)\times n} B_{p\times q} = O_{(m-q)\times q}$, if $n \leq p, m > q$;

where for $i = q+1, q+2, \dots, m$; R_i is the i^{th} row of $A_{m\times n}$.

(iii). $A_{m\times p}B_{p\times m} = I_m$ and $A_{m\times n}\left(B_{m+1}, B_{m+2}, \dots, B_q\right)_{p\times(q-m)} = O_{m\times(q-m)}$, if $n > p, m < q$.

(iv). $A_{q\times p}B_{p\times q} = I_q$ and $\begin{pmatrix} R_{q+1} \\ R_{q+2} \\ \dots\dots \\ \dots\dots \\ R_m \end{pmatrix}_{(m-q)\times n} B_{p\times q} = O_{(m-q)\times q}$, if $n > p, m > q$.

Theorem (1.5) [3]

For a given nonzero matrix $A_{m\times n}$ in $M(F)$, if there exists $B_{p\times q}$ in $M(F)$ such that $A_{m\times n}B_{p\times q} = I_{m\times q}$, then $m \leq n$, except the case $m > n > q$.

Theorem (1.6) [3]

For a given nonzero matrix $B_{p\times q}$ in $M(F)$, if there exists $A_{m\times n}$ in $M\ (F)$ such that

$A_{m\times n}B_{p\times q} = I_{m\times q}$ then $q \leq n$, except the case $q > n > m$.

Again, we have the following properties in [2] and [5].

Theorem (1.7) [2]

For any two matrices $A_{m\times n}, B_{p\times q} \in M(F)$, $\left(A_{m\times n} + B_{p\times q}\right)^T = A^T_{m\times n} + B^T_{p\times q}$

Theorem (1.8) [2]

For any two matrices $A_{m\times n}, B_{p\times q} \in M(F)$, $\left(A_{m\times n} B_{p\times q}\right)^T = B^T_{p\times q} A^T_{m\times n}$

Definition (1.6) [2]

Let $A_{m\times n} = \left(a_{ij}\right)_{m\times n} \in M(F)$. If $m \le n$, then for $i = 1,2,\dots,m$;

$j = i, i+1, \dots, i+(n-m)$, $a_{ij}'s$ are called the diagonal elements of $A_{m\times n}$.

If $m > n$, then for $j = 1,2,\dots,n;\ i = j, j+1, \dots, j+(m-n)$, $a_{ij}'s$ are called the diagonal elements of $A_{m\times n}$.

In each case, the portion of $A_{m\times n}$, formed by these diagonal elements is called the diagonal of $A_{m\times n}$.

All the elements of $A_{m\times n}$, other than the diagonal elements, are called the non-diagonal elements of $A_{m\times n}$.

Definition (1.7) [2] (a)

(a) A matrix $A_{m\times n} = \left(a_{ij}\right)_{m\times n} \in M(F)$ is said to be symmetric if, when $m \le n$, then for $i = 2,3,\dots,m;\ j = 1,2,\dots.,m-1$, if $i > j$ then $a_{ij} = a_{j(i+n-m)}$ and when $\ge n$, then for $j = 2,3,\dots,n;\ i = 1,2,\dots.,n-1$, if $i < j$ then $a_{ij} = a_{(j+m-n)i}$.

(b) A matrix $A_{m\times n} = \left(a_{ij}\right)_{m\times n} \in M(F)$ is said to be skew-symmetric if all the diagonal elements of $A_{m\times n}$ are zero and when $m \le n$, then for $i = 2,3,\dots,m$; $j = 1,2,\dots.,m-1$, if $i > j$ then $a_{ij} = -a_{j(i+n-m)}$ and when $m \ge n$, then for $j = 2,3,\dots,n$; $i = 1,2,\dots.,n-1$, if $i < j$ then $a_{ij} = -a_{(j+m-n)i}$.

(c) A matrix $A_{m\times n} = \left(a_{ij}\right)_{m\times n} \in M(F)$ is said to be weak skew-symmetric if, when $\le n$, then for $i = 2,3,\dots,m$; $j = 1,2,\dots.,m-1$, if $i > j$ then $a_{ij} = -a_{j(i+n-m)}$ and when $m \ge n$, then for $j = 2,3,\dots,n$; $j = 1,2,\dots.,n-1$, if $i < j$ then $a_{ij} = -a_{(j+m-n)i}$.

Example (1.2)

In the real matrices $A = \begin{pmatrix} \mathbf{2} & \mathbf{1} & 0 & -1 \\ 0 & \mathbf{5} & \mathbf{2} & 3 \\ -1 & 3 & \mathbf{-5} & \mathbf{7} \end{pmatrix}$, $B = \begin{pmatrix} \mathbf{0} & 1 & -3 \\ \mathbf{0} & \mathbf{0} & 4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -1 & \mathbf{0} & \mathbf{0} \\ 3 & -4 & \mathbf{0} \end{pmatrix}$, $C = \begin{pmatrix} \mathbf{2} & 1 & -3 \\ \mathbf{0} & \mathbf{5} & 4 \\ \mathbf{1} & \mathbf{0} & \mathbf{8} \\ -1 & \mathbf{-8} & \mathbf{9} \\ 3 & -4 & \mathbf{1} \end{pmatrix}$, the bold elements are diagonal elements and the non-bold elements are non-diagonal elements. Also, A is symmetric, *B* is skew-symmetric and *C* is weak skew-symmetric.

Theorem (1.9) [2]

Transpose of a symmetric, skew-symmetric and a weak skew-symmetric matrix is symmetric, skew-symmetric and weak skew-symmetric respectively.

Theorem (1.10) [2]

For any two matrices $A, B \in M$ (),

(i). If A, B be symmetric, then the matrix $A + B$ is not, in general, symmetric.

(ii). If A, B be skew-symmetric, then the matrix $A + B$ is not, in general, skew-symmetric.

(iii). If A, B be weak skew-symmetric, then the matrix $A + B$ is not, in general, weak skew-symmetric.

(iv). if A, B be two symmetric (or skew-symmetric) matrices of the same order such that $AB = BA$, then AB may not be a symmetric matrix (or skew-symmetric).

Theorem (1.11) [2]

For any square symmetric matrix A and any matrix P in $M\ (F)$, $P^T AP$ is a symmetric matrix; but the result fails to be hold good if A be a non-square symmetric matrix.

Note (1.2) [2]

Let $A_{m\times n} \in M(F)$ with $\neq n$. Then obviously

$A_{m\times n} \neq \frac{1}{2}(A_{m\times n} + (A_{m\times n})^T) + \frac{1}{2}(A_{m\times n} - (A_{m\times n})^T)$, as the matrix on the right hand side is of order $\max\{m, n\} \times \max\{m, n\} \neq (m, n)$, provided Char($F$) ≠ 2. Therefore a natural question arises. Is it possible to express a matrix over a field as a sum of a symmetric matrix and a skew-symmetric matrix? The answer to this question is affirmative, given by means of the following proposition.

Theorem (1.12) [2]

Every matrix over a field F can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix, provided Char (F) ≠ 2; but the expression is not, in general, unique.

Proof

Let $A = (a_{ij})_{m\times n} \in M\ (F)$ be arbitrary. If $m = n$, then the result is obvious, as

$A_{m\times n} = \frac{1}{2}(A_{m\times n} + (A_{m\times n})^T) + \frac{1}{2}(A_{m\times n} - (A_{m\times n})^T)$. (Since Char ($F$) ≠ 2, hence 2^{-1} Exists in F).

Let $m < n$. Let $B = (b_{ij})_{m\times n}$ be a symmetric matrix and $C = (c_{ij})_{m\times n}$ be a skew-symmetric matrix in $M(F)$ such that $= B + C$, i.e., $(a_{ij})_{m\times n} = (b_{ij})_{m\times n} + (c_{ij})_{m\times n}$ **(1)**

Then for $= 1, 2, \dots, m;\ j = 1, 2, \dots, n$, $b_{ij} + c_{ij} = a_{ij}$ **(2)**

Since $m < n$ and B, C are symmetric and skew-symmetric matrices, respectively, hence the diagonal elements of B and A are same (since the diagonal elements of C are zero). Thus the diagonal elements of B are determined.

Now for the non-diagonal elements of B and C we have for $i = 2, 3, \dots, m;\ j = 1, 2, \dots, m - 1$, if $i > j$,

then $b_{ij} = b_{j(i+n-m)}$ **(3)**

and $c_{ij} = -c_{j(i+n-m)}$ **(4)**

From (2) we have for $= 2, 3, \ldots, m$; $j = 1, 2, \ldots, m-1$, if $i > j$, then $b_{ij} + c_{ij} = a_{ij}$ **(5)**

And $b_{j\,(i+n-m)} + c_{j(i+n-m)} = a_{j(i+n-m)}$, i.e., $b_{ij} - c_{ij} = a_{j(i+n-m)}$ **(6)**

(by (3), (4)).

From (5) and (6) we get

$i = 2, 3, \ldots, m$; $j = 1, 2, \ldots, m-1$, if $> j$, then $b_{ij} = 2^{-1}\left(a_{ij} + a_{j(i+n-m)}\right)$ **(7)**

$c_{ij} = 2^{-1}\left(a_{ij} - a_{j(i+n-m)}\right)$ **(8)**

(since Char (F) $\neq 2$, hence 2^{-1} exists in F).

$b_{j(i+n-m)} = b_{ij} = 2^{-1}\left(a_{ij} + a_{j(i+n-m)}\right)$ **(9)**

(by (7))

$c_{j(i+n-m)} = -c_{ij} = -2^{-1}\left(a_{ij} - a_{j(i+n-m)}\right)$ **(10)**

(by (8))

Thus B and C are determined.

If $m > n$, then similarly, we have $\left(\left(a_{ij}\right)_{m\times n}\right)^T = E_{n\times m} + F_{n\times m}$ **(11)**

where $E_{n\times m}$ is a symmetric matrix and $F_{n\times m}$ is a skew-symmetric matrix.

From (11), we get $A = \left(a_{ij}\right)_{m\times n} = (E_{n\times m} + F_{n\times m})^T = (E_{n\times m})^T + (F_{n\times m})^T$ **(12)**

Since $E_{n\times m}$ is a symmetric matrix and $F_{n\times m}$ is a skew-symmetric matrix, hence $(E_{n\times m})^T$ is a symmetric matrix and $(F_{n\times m})^T$ is a skew-symmetric matrix. Hence the result

To prove that the expression is not unique, it is sufficient to consider some examples. Consider the matrix

$A = \begin{pmatrix} 1 & 1 & 3 & 4 \\ 4 & 2 & 1 & 3 \\ -3 & 0 & 4 & 0 \\ 2 & 1 & -2 & 0 \end{pmatrix}$. Then $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$ **(13)**

and $\frac{1}{2}(A + A^T)$ is symmetric and $\frac{1}{2}(A - A^T)$ is skew-symmetric .

Again, we see that $A = \begin{pmatrix} 1 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 4 \\ -1 & 0 & 0 & 3 \\ -4 & -3 & 0 & 0 \end{pmatrix}$ **(14)**

And the first matrix of right hand side of (14) is symmetric and the second one is skew-symmetric.

Clearly the expressions (13) and (14) are distinct.

Again, consider another example in which

$$A=\begin{pmatrix}2&1&-1&-1&5\\-6&3&2&5&0\\4&-1&3&2&6\end{pmatrix}\text{. Then } A=\begin{pmatrix}2&1&-1&-\frac{7}{2}&\frac{9}{2}\\-\frac{7}{2}&3&2&5&-\frac{1}{2}\\\frac{9}{2}&-\frac{1}{2}&3&2&6\end{pmatrix}+\begin{pmatrix}0&0&0&\frac{5}{2}&\frac{1}{2}\\-\frac{5}{2}&0&0&0&\frac{1}{2}\\-\frac{1}{2}&-\frac{1}{2}&0&0&0\end{pmatrix} \quad \textbf{(15)}$$

And the first matrix of right hand side of (15) is symmetric and the second one is skew-symmetric.

$$\text{Again } A=\begin{pmatrix}2&1&-5&-2&5\\-2&3&2&4&0\\5&0&3&2&6\end{pmatrix}+\begin{pmatrix}0&0&4&1\\-4&0&0&1\\-1&-1&0&0\end{pmatrix} \quad \textbf{(16)}$$

And the first matrix of right hand side of (16) is symmetric and the second one is skew-symmetric.

Clearly the expressions (15) and (16) are distinct.

Note (1.3) [2]

From the proof of theorem (1.12) we observed that any $m\times n$ matrix over a field F can be expressed as the sum of a symmetric and a skew-symmetric matrix of the same order. Therefore a natural question arises. Is it always possible to express a matrix $A_{m\times n}$ as $A_{m\times n}=B_{p\times q}+C_{r\times s}$, where m,n,p,q,r,s are given positive integers such that $m=\max\ \{p,r\}$ and $n=\max\ \{q\ ,s\}$ and $B_{p\times q}$ is symmetric and $C_{r\times s}$ is skew-symmetric and $(p,q)\neq(r,s)$? The answer to this question is given by means of the following proposition.

Again from theorem (1.12) it is observed that, if we wish to express $A_{m\times n}$ as $A_{m\times n}=B_{m\times n}+C_{m\times n}$ only, where $B_{m\times n}$ is symmetric and $C_{m\times n}$ is skew-symmetric, then this expression unique.

Theorem (1.13) [2]

Let $A_{m\times n}\in M(F)$ and p,q,r,s be positive integers such that$m=\max\{p\ ,r\}$,

$n=\max\{q\ ,s\}$ and $(p,q)\neq(r,s)$. Then $A_{m\times n}$ can be expressed as sum of a symmetric matrix of order $p\times q$ and a skew-symmetric matrix of order $r\times s$ if (i) $p\geq q\geq s\geq r$ or (ii) $q\geq p\geq r\geq s$ or

(iii). $p\geq q\geq s\ ,p\geq r\geq s$ or (iv) $q\geq p\geq r\ ,q\geq s\geq r$.

And this expression is not possible, in general, in other cases, i.e., if (v) $p\geq r\geq s\geq q$ or

(vi). $q\geq s\geq r\geq p$ or (vii) r$\geq p\geq q\geq s$ or (viii) $s\geq q\geq p\geq r$ or

(ix). $p\geq q,p\geq r,s\geq q,s\geq r$ or (x) $\geq p,q\geq s,r\geq p,r\geq s$.

Definition (1.8) [5]

A matrix $A_{m\times n}\in M(F)$ is said to be an idempotent matrix if $A^2_{m\times n}=A_{m\times n}$.

Theorem (1.14) [5]

Let $A_{m\times n}\in M(F)$ be an idempotent matrix. Then $I_{p\times q}-A_{m\times n}$ is idempotent iff $p,q\ \geq m$ and $,q\ \geq n$.

Theorem (1.15) [5]

Let $A_{m\times n},B_{p\times q}\in M(F)$ such that $A_{m\times n}+B_{p\times q}=I_{r\times s}$ and $A_{m\times n}B_{p\times q}=O_{m\times q}$, where $r=\max\{m,p\}\,,s=\max\ \{n,q\}$. Then $A_{m\times n}$ and $B_{p\times q}$ both are idempotent if $n=p\geq m,q$.

Theorem (1.16) [5]

If $m, n, p, q \in \mathbb{N}$ such that $p, q \geq m$ and $p, q \geq n$ and if $A_{m\times n} \in M(F)$ be an idempotent matrix, then $\forall k \in \mathbb{N}$, $\left(I_{p\times q} + A_{m\times n}\right)^k = I_{p\times q} + (2^k - 1)A_{m\times n}$.

Definition (1.9) [5]

The trace of a matrix $A_{m\times n} \in M(F)$, denoted by $trA_{m\times n}$, is defined as the sum of all diagonal elements of $A_{m\times n}$.

Theorem (1.17) [5]

For any $A_{m\times n} \in M\ (F)$, $trA_{m\times n} = trA^t_{m\times n}$.

Theorem (1.18) [5]

For any two matrices $A_{m\times n}, B_{p\times q} \in M(F)$,

(i) $tr\left(A_{m\times n} + B_{p\times q}\right) \neq trA_{m\times n} + trB_{p\times q}$, in general.

(ii) $tr\left(A_{m\times n}B_{p\times q}\right) \neq tr\left(B_{p\times q}A_{m\times n}\right)$, in general.

Again in [4], it is stated that, for a given $A_{m\times n} \in M\ (F)$, if there exists $B_{m\times n} \in M(F)$, such that $A_{m\times n}B_{m\times n} = B_{m\times n}A_{m\times n} = I_{m\times n}$, then $B_{m\times n}$ is unique and we call $B_{m\times n}$ as the inverse of $A_{m\times n}$, and denoted by $A^{-1}_{m\times n}$.

$$\text{Clearly } B_{m\times n} = A^{-1}_{m\times n} = \begin{cases} \left(A^{-1}_{m\times m}\ O_{m\times(n-m)}\right), if\ m < n \\ \begin{pmatrix} A^{-1}_{n} \\ O_{(m-n)\times n} \end{pmatrix}, if\ m > n \\ A^{-1}_{m}, if\ m = n \end{cases}$$

Again in [4] Sherman-Morrison Rank One Update Formula and Sherman-Morrison-Woodbury Formula [8] in traditional matrix algebra are extended in extended matrix algebra on $M(F)$.

Theorem (1.19) [4]

Let $m, n \in \mathbb{N}$ with $\mathrm{m} < n$ and

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \dots. \\ \dots. \\ c_m \end{pmatrix}_{m\times 1}, D = \begin{pmatrix} d_1 & d_2 & \dots\dots & d_m & 0 & 0 & \dots.. & 0 \end{pmatrix}_{1\times n} \in M(F).$$

Then $(I_{m\times n} + CD)^{-1} = I_{m\times n} - \frac{CD}{1+DC}$, provided $(I_{m\times n} + CD)^{-1}$ exists and $1 + DC \neq 0$.

Corollary (1.19)[4]: Let $m, n \in \mathbb{N}$ with $\mathrm{m} > n$ and

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \dots. \\ \dots. \\ c_n \\ 0 \\ 0 \\ \dots:: \\ 0 \end{pmatrix}_{m\times 1}, D = \begin{pmatrix} d_1 & d_2 & \dots.. & d_n \end{pmatrix}_{1\times n} ..$$

Then $(I_{m\times n}+CD)^{-1}=I_{m\times n}-\frac{CD}{1+DC}$, provided $(I_{m\times n}+CD)^{-1}$ exists and $1+DC\neq 0$.

Theorem (1.20) [4]

Let $m,n\in\mathbb{N}$ with m $<n$ and $A_{m\times n}=\left(\left(a_{ij}\right)_{m\times m}\quad O_{m\times(n-m)}\right)$,

$C=\begin{pmatrix}c_1\\c_2\\\cdots\\\cdots\\c_m\end{pmatrix}_{m\times 1}$, $D=(d_1\quad d_2\quad\ldots\ldots\quad d_m\quad 0\quad 0\quad\ldots\ldots\quad 0)_{1\times n}\in M(F)$.

Then $(A_{m\times n}+CD)^{-1}=A_{m\times n}^{-1}-\frac{A_{m\times n}^{-1}CDA_{m\times n}^{-1}}{1+DA_{m\times n}^{-1}C}$, provided $(A_{m\times n}+CD)^{-1}$ and $A_{m\times n}^{-1}$ both exist and $1+DA_{m\times n}^{-1}C\neq 0$.

Corollary (1.20): Let $m,n\in\mathbb{N}$ with m $>n$ and $A_{m\times n}=\begin{pmatrix}\left(a_{ij}\right)_{n\times n}\\O_{(m-n)\times n}\end{pmatrix}$, $C=\begin{pmatrix}c_1\\c_2\\\cdots\\\cdots\\c_n\\0\\\cdots\\\cdots\\0\end{pmatrix}_{m\times 1}$,

$D=(d_1\quad d_2\quad\ldots\ldots\quad d_n)_{1\times n}\in M(F)$.

Then $(A_{m\times n}+CD)^{-1}=A_{m\times n}^{-1}-\frac{A_{m\times n}^{-1}CDA_{m\times n}^{-1}}{1+DA_{m\times n}^{-1}C}$, provided $(A_{m\times n}+CD)^{-1}$ and $A_{m\times n}^{-1}$ both exist and

$1+DA_{m\times n}^{-1}C\neq 0$.

Theorem (1.21) [4]

Let $m,n,k\in\mathbb{N}$ with m $<n$ and $A_{m\times n}=\left(\left(a_{ij}\right)_{m\times m}\quad O_{m\times(n-m)}\right)$,

$C=\left(c_{ij}\right)_{m\times k}$, $D=\left(\left(d_{ij}\right)_{k\times m}\quad O_{k\times(n-m)}\right)_{k\times n}\in M(F)$.

Then $(A_{m\times n}+CD)^{-1}=A_{m\times n}^{-1}-A_{m\times n}^{-1}C(I_k+DA_{m\times n}^{-1}C)^{-1}DA_{m\times n}^{-1}$, provided $(A_{m\times n}+CD)^{-1}$, $A_{m\times n}^{-1}$ and $(I_k+DA_{m\times n}^{-1}C)^{-1}$ exist .

Corollary (1.21) [4]: Let $m,n,k\in\mathbb{N}$ with m $>n$ and $A_{m\times n}=\begin{pmatrix}\left(a_{ij}\right)_{n\times n}\\O_{(m-n)\times n}\end{pmatrix}$, $C=\begin{pmatrix}\left(c_{ij}\right)_{n\times k}\\O_{(m-n)\times k}\end{pmatrix}$,

$D=\left(d_{ij}\right)_{k\times n}\in M(F)$. Then $(A_{m\times n}+CD)^{-1}=A_{m\times n}^{-1}-A_{m\times n}^{-1}C(I_k+DA_{m\times n}^{-1}C)^{-1}DA_{m\times n}^{-1}$, provided $(A_{m\times n}+CD)^{-1}$, $A_{m\times n}^{-1}$ and $(I_k+DA_{m\times n}^{-1}C)^{-1}$ exist

2. MAIN RESULTS

Here we shall study some more properties of extended matrix algebra.

Theorem (2.1)

For any three matrices $A_{m\times n}, B_{m\times n}, C_{n\times m}\in M(F)$,

(i) $tr(A_{m\times n}+B_{m\times n})=trA_{m\times n}+trB_{m\times n}$.

(ii) $tr(A_{m\times n}B_{m\times n}) \neq tr(B_{m\times n}A_{m\times n})$, in general.

(iii) $tr(A_{m\times n}C_{n\times m}) = tr(C_{n\times m}A_{m\times n})$.

(iv) $tr(A_{m\times n} + C_{n\times m}) \neq trA_{m\times n} + trC_{n\times m}$, in general.

Proof

Let $A_{m\times n} = \left(a_{ij}\right)_{m\times n}, B_{m\times n} = \left(b_{ij}\right)_{m\times n}, C_{n\times m} = \left(c_{ij}\right)_{m\times n}$.

(i) We have $A_{m\times n} + B_{m\times n} = \left(d_{ij}\right)_{m\times n}$, where

for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, $d_{ij} = a_{ij} + b_{ij}$ **(17)**

From (1) it is clear that the diagonal elements of $A_{m\times n} + B_{m\times n}$ are the sum of the corresponding diagonal elements of $A_{m\times n}$ and $B_{m\times n}$.

Hence $tr(A_{m\times n} + B_{m\times n}) = trA_{m\times n} + trB_{m\times n}$.

(ii) Consider the real matrices $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -2 & 2 & 1 & 0 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ and $= \begin{pmatrix} 2 & 3 & 1 & 0 \\ 3 & 0 & -2 & 2 \\ -1 & 2 & 4 & 3 \end{pmatrix}$.

Then $AB = \begin{pmatrix} 3 & 1 & -3 & -3 \\ 1 & -4 & -2 & 7 \\ 7 & 13 & 9 & 8 \end{pmatrix}$ and $= \begin{pmatrix} -1 & 7 & 3 & 8 \\ -3 & -2 & -7 & -2 \\ 7 & 8 & 11 & 14 \end{pmatrix}$.

Now $trAB = (3 - 4 + 9) + (1 - 2 + 8) = 15$ and $trBA = (-1 - 2 + 11) + (7 - 7 + 14) = 22$.

Therefore *trAB* ≠ *trBA*. Hence the result

(iii) We have $A_{m\times n}C_{n\times m} = \left(e_{ij}\right)_{m\times m}$, where

For $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, m$, $e_{ij} = \sum_{k=i}^{n} a_{ik}c_{kj}$ **(18)**

From (2) it is clear that the diagonal elements of $A_{m\times n}C_{n\times m}$ are given by

for $= 1, 2, \ldots, m$, $e_{ii} = \sum_{k=i}^{n} a_{ik}c_{ki}$ **(19)**

Again $C_{n\times m}A_{m\times n} = \left(f_{ij}\right)_{n\times n}$, where

For $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$, $f_{ij} = \sum_{k=i}^{m} c_{ik}a_{kj}$ **(20)**

From (4) it is clear that the diagonal elements of $C_{n\times m}A_{m\times n}$ are given by

For $= 1, 2, \ldots, n, f_{ii} = \sum_{k=i}^{m} c_{ik}a_{ki}$ **(21)**

Now $tr(A_{m\times n}C_{n\times m}) = \sum_{i=1}^{m} e_{ii} = \sum_{i=1}^{m}\sum_{k=i}^{n} a_{ik}c_{ki}$ **(22)**

(by (19)) .

And $tr(C_{n\times m}A_{m\times n}) = \sum_{i=1}^{n} f_{ii} = \sum_{i=1}^{n}\sum_{k=i}^{m} c_{ik}a_{ki}$ (by (21))

$= \sum_{k=1}^{n}\sum_{i=i}^{m} c_{ki}a_{ik}$ (replacing the indices *i* , *k* by *k, i* respectively)

$= \sum_{i=1}^{m}\sum_{k=i}^{n} a_{ik}c_{ki} = tr(A_{m\times n}C_{n\times m})$ (by (22)) .

Therefore $(A_{m\times n}C_{n\times m}) = tr(C_{n\times m}A_{m\times n})$.

(iv) Consider the real matrices $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ -2 & 2 & 1 & 0 \\ 3 & 1 & 2 & 4 \end{pmatrix}$ and $= \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 4 \\ 5 & 2 & 1 \\ 2 & 4 & 3 \end{pmatrix}$.

Then $+C = \begin{pmatrix} 3 & 1 & 2 & 2 \\ -1 & 5 & 5 & 0 \\ 8 & 3 & 3 & 4 \\ 2 & 4 & 3 & 0 \end{pmatrix}$.

Now $trA = (1+2+2)+(0+1+4) = 10$, $trC = (2+3+1)+(1+2+3) = 12$ and

$tr(A+C) = 3+5+3+0 = 11 \neq 22 = trA + trC$. Hence the result

Definition (2.1)

A matrix $A_{m\times n} \in M(F)$ is said to be a diagonal matrix if its non-diagonal elements are all zero.

Theorem (2.2)

If $A, B \in M(F)$ be of the same order such that A is a diagonal matrix with no two diagonal elements are equal, and B commutes with A, then B may not be a diagonal matrix.

Proof

Consider the real matrix $A = \begin{pmatrix} -4 & 4 & -1 & 0 & 0 \\ 0 & -3 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 & 2 \end{pmatrix}$. Clearly, all the diagonal elements of A are distinct.

And, consider the real matrix $B = \begin{pmatrix} \frac{68}{13} & \frac{-12}{13} & -1 & -4 & 4 \\ 0 & 5 & -3 & -4 & 4 \\ 0 & 0 & 1 & -2 & 2 \end{pmatrix}$ which is clearly not a diagonal matrix.

Now $AB = \begin{pmatrix} \frac{-272}{13} & \frac{308}{13} & -9 & 2 & -2 \\ 0 & -15 & 12 & 6 & -6 \\ 0 & 0 & 1 & -2 & 2 \end{pmatrix}$ and $BA = \begin{pmatrix} \frac{-272}{13} & \frac{308}{13} & -9 & 2 & -2 \\ 0 & -15 & 12 & 6 & -6 \\ 0 & 0 & 1 & -2 & 2 \end{pmatrix}$ so that $AB = BA$.

We know that, for any positive integral power of a square symmetric matrix is symmetric; any positive even integral power of a square skew-symmetric matrix is symmetric and any positive odd integral power of a square skew-symmetric matrix is skew-symmetric. Let us see whether this result holds good in extended matrix algebra.

Theorem (2.3)

(i) Any positive integral power of a symmetric matrix in $M\ (F)$ may not be symmetric.

(ii) Any positive even integral power of a skew-symmetric matrix in $M\ (F)$ may not be symmetric and positive odd integral power of a skew-symmetric matrix in $M\ (F)$ may not be skewed-symmetric.

Proof

(i) For example, consider the real symmetric matrix $= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & -1 & 2 & 0 & 3 \\ 5 & 3 & 1 & -2 & 4 \end{pmatrix}$

Then $A^2 = \begin{pmatrix} 24 & 9 & 10 & -2 & 23 \\ 10 & 15 & 12 & 12 & 25 \\ 22 & 10 & 22 & 18 & 38 \end{pmatrix}$ Which is not symmetric.

(ii) Consider the real skew-symmetric matrix $= \begin{pmatrix} 0 & 0 & 3 & 4 & 5 \\ -3 & 0 & 0 & -6 & 7 \\ -4 & 6 & 0 & 0 & -8 \\ -5 & -7 & 8 & 0 & 0 \end{pmatrix}$.

Then $B^2 = \begin{pmatrix} -32 & -10 & 32 & 0 & -24 \\ 30 & 42 & -57 & -12 & -15 \\ -18 & 0 & -12 & -52 & 22 \\ -11 & 48 & -15 & 22 & -138 \end{pmatrix}$ which is not symmetric.

Again $B^3 = \begin{pmatrix} -98 & 192 & -96 & -68 & -486 \\ 162 & -258 & -6 & -132 & 900 \\ 308 & 292 & -470 & -72 & 6 \\ -194 & -244 & 143 & -332 & 371 \end{pmatrix}$ which is not skew-symmetric.

Definition (2.2)

A matrix $A_{m\times n} \in M(F)$ is said to be involuntary matrix if $A^2_{m\times n} = I_{m\times n}$.

We know that for any two matrices $A_{m\times n}, B_{n\times m} \in M\ (F)$, the matrix

$L = \begin{pmatrix} I_n - B_{n\times m}A_{m\times n} & B_{n\times m} \\ 2A_{m\times n} - A_{m\times n}B_{n\times m}A_{m\times n} & A_{m\times n}B_{n\times m} - I_m \end{pmatrix}$ is involuntary. In theorem (2.4) we shall study about this property in extended matrix algebra.

Theorem (2.4)

Let $A_{m\times n}, B_{p\times q} \in M(F)$. Then the matrix

$L = \begin{pmatrix} I_{p\times n} - B_{p\times q}A_{m\times n} & B_{p\times q} \\ 2A_{m\times n} - A_{m\times n}B_{p\times q}A_{m\times n} & A_{m\times n}B_{p\times q} - I_{m\times q} \end{pmatrix}$ may not be involuntary.

Proof

Consider two real matrices$A_{3\times 3} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 3 & -1 \\ 0 & -2 & 1 \end{pmatrix}, B_{4\times 3} = \begin{pmatrix} 1 & 2 & 3 \\ -3 & 2 & 0 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$.

Then it can be easily verified that $L_{7\times 6} = \begin{pmatrix} I_{4\times 3} - B_{4\times 3}A_{3\times 3} & B_{4\times 3} \\ 2A_{3\times 3} - A_{3\times 3}B_{4\times 3}A_{3\times 3} & A_{3\times 3}B_{4\times 3} - I_{3\times 3} \end{pmatrix}$

$= \begin{pmatrix} -2 & 0 & -2 & 1 & 2 & 3 \\ -7 & -5 & 5 & -3 & 2 & 0 \\ 3 & 7 & -3 & 1 & -1 & 2 \\ -1 & -4 & -2 & 3 & 2 & 1 \\ 4 & 7 & 0 & -1 & -3 & -1 \\ -26 & -19 & 13 & -8 & 10 & 4 \\ 17 & 15 & -12 & 7 & -5 & 1 \end{pmatrix}$ is not involuntary.

Note (2.1)

We know in traditional matrix algebra that, if a square matrix *A* over a field *F* of characteristic zero, be an orthogonal matrix, *i.e.,* if $AA^T = I$, then$AA^T = A^TA = I$, and the set of row vectors and the set of column vectors of A are orthogonal sets. Here we introduce the definition (2.3) of right orthogonal and left orthogonal matrices and the in theorem (2.5) we study about properties of orthogonal matrices in extended matrix algebra.

Definition (2.3)

A matrix $A_{m\times n} \in M\ (F)$ is said to be right orthogonal I $A_{m\times n}A^T_{m\times n} = I_{m\times m}$. $A_{m\times n}$ is said to be left orthogonal if $A^T_{m\times n}A_{m\times n} = I_{n\times n}$.

Theorem (2.5)

(i) A right orthogonal matrix in $M\ (F)$ may not be left orthogonal and a left orthogonal matrix in $M\ (F)$ may not be right orthogonal.

(ii) If a matrix $A_{m\times n} \in M\ (F)$ be right orthogonal, then the set of row vectors of $A_{m\times n}$ is an orthogonal set of vectors over the field *F*. And if $A_{m\times n}$ be left orthogonal, then the set of column vectors of $A_{m\times n}$ is an orthogonal set of vectors over the field *F*.

(iii) If a matrix $A_{m\times n} \in M\ (F)$ be right orthogonal as well as left orthogonal, then $m = n$ and $A_{m\times n}$ is an orthogonal matrix; and the set of row vectors as well as the set of column vectors of $A_{m\times n}$ are orthogonal sets of vectors over the field *F* .

(iv) If a matrix $A_{m\times n} \in M(F)$ be right orthogonal then it is a regular element in the weak hemi-ring$(M(F), +, .)$; and if it is left orthogonal then it is a regular element in the weak hemi-ring$(M(F), +, .)$.

Proof

Trivial.

Note (2.2)

We know in traditional matrix algebra that, given a system of *n* linear equations in *n* unknowns such that the coefficient matrix is non-singular, then the system is consistent and has a unique solution. Also, we know that, given a system of *m* linear equations in *n* unknowns such that the rank of the coefficient matrix is equal to the rank of the augmented matrix, then the system is consistent. In theorem (2.6) and corollary (2.6) we shall study about this property in extended matrix algebra.

Theorem (2.6)

Let $A_{m\times n} = \left(a_{ij}\right)_{m\times n}$, $B_{m\times 1} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_m \end{pmatrix} \in M\ (F)$ and $X_{p\times 1} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_p \end{pmatrix}$ be an unknown matrix over the field *F* and char(*F*) = 0 . Also, let there exists $C_{r\times s} = \left(c_{ij}\right)_{r\times s} \in M(F)$ such that $C_{r\times s}A_{m\times n} = I_{r\times n}$.

Let us consider the system of linear equations $A_{m\times n}X_{p\times 1} = B_{m\times 1}$ **(23)**

Then **(i)** (23) is consistent and has a unique solution if either $r = n = p$ or $p = r < n$.

(ii) (23) is consistent and has many solutions if either $r = n < p$ or $r < n = p$ or $r < n < p$ or $r < p < n$.

(iii) (23) is consistent and has a unique solution if either $r = n > p$ or $p < r < n$ or $r > n = p$ or $r > n > p$,

provided in each of the cases, $C_{r\times s}B_{m\times 1}$ is of the form $\begin{pmatrix} d_1 \\ d_2 \\ \cdots \\ \cdots \\ d_p \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{r\times 1}$; otherwise (23) is inconsistent.

(iv) (23) is consistent and has many solutions if either $n < r < p$ or $n < p < r$ or $p = r > n$; provided in each of the cases, $C_{r\times s}B_{m\times 1}$ is of the form $\begin{pmatrix} d_1 \\ d_2 \\ \cdots \\ \cdots \\ d_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{r\times 1}$; otherwise (23) is inconsistent.

Proof

From (23) we get $C_{r\times s}\{A_{m\times n}X_{p\times 1}\} = C_{r\times s}B_{m\times 1}$. This implies that $\{C_{r\times s}A_{m\times n}\}X_{p\times 1} = D_{r\times 1}$ … **(24)**

where $D_{r\times 1} = \begin{pmatrix} d_1 \\ d_2 \\ \cdots \\ \cdots \\ d_r \end{pmatrix}$ and for $i = 1, 2, \ldots, r$, $d_i = \sum_{k=1}^{\min\{s,m\}} c_{ik}b_k$.

Now considering several cases, the remaining part of the proof is just now routine check.

Corollary (2.6): Let $A_{m\times n} = (a_{ij})_{m\times n}$, $B_{m\times 1} = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_m \end{pmatrix} \in M\,(F)$ and $X_{p\times 1} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_p \end{pmatrix}$ be an unknown matrix over the field F and char(F) = 0 . Then the system of linear equations $A_{m\times n}X_{p\times 1} = B_{m\times 1}$ is consistent iff rank of $A_{m\times n}$ = rank of $\bar{A}_{m\times n}\ (= (A_{m\times n}\ B_{m\times 1}))$.

Definition (2.4)

$A_{m\times n} \in M(F)$ and char(F) = 0. Let $X_{p\times 1}$ be an unknown vector. If there exists $\lambda \in F$ such that the system of equations $(A_{m\times n} - \lambda I_{m\times n})X_{p\times 1} = O_{m\times 1}$ has a non-zero solution in $M(F)$, then λ is called an eigen

value of $A_{m\times n}$ and corresponding non-zero vector in $M(F)$, obtained by solving this system of equations, is called an eigen vector of $A_{m\times n}$ corresponding to eigen value λ.

Theorem (2.7)

$A_{m\times n} \in M(F)$ and char (F) = 0. Let $\lambda\ (\in F)$ be an eigen value of $A_{m\times n}$. Then

(i) At least $m - n + 1$ rows of$(A_{m\times n} - \lambda I_{m\times n})$ are redundant, provided $m > n$.

(ii) At least $n - m + 1$ columns of$(A_{m\times n} - \lambda I_{m\times n})$ are redundant, provided $m < n$.

Proof

Trivial.

CONCLUSIONS

Further study regarding more properties of extended algebra may be continued.

REFERENCES

1. Jayanta Biswas, *Extension of Matrix Algebra*, Oral Presentation, The 9th International IMBIC Conference MSAST 2015, Kolkata, December 21-23, 2015; Abstracts, Volume 4, Page-279, ISBN 978-3-5, 2015.
2. Jaya Biswas, *Some properties of Extended Matrix Algebra,* Oral Presentation, National Seminar on Recent Developments In Mathematics And Its Applications (NSRDMA-2016), Department of Mathematics, University of Kalyani, January 21-22, 2016; Abstracts, Page-40-41.
3. Jayanta Biswas, *Extension of Matrix Algebra and Linear spaces of Linear Transformations,* International Journal of Novelty Research in Physics, Chemistry and Mathematics, ISSN-2394-9651, Vol-3, Issue 1, pp: 29-42, 2016.
4. Jayanta Biswas, *Sherman-Morrison Rank One Update Formula and Sherman-Morrison-Woodbury Formula in Extended Matrix Algebra,* Oral Presentation, National Seminar On Recent Advances In Mathematics And Its Applications (RAMA-2016), Department of Pure Mathematics, University of Calcutta, February 24-25, 2016; Abstracts, Page 31-32.
5. Jayanta Biswas, *Some More properties of Extended Matrix Algebra,* Oral Presentation, National Seminar on Analysis And Applications: Celebrating 100 years of the General Theory of Relativity, Department of Mathematics, West Bengal State University, March 10-11, 2016; Abstracts, Page- 17-18.
6. K. R. Matthews, *Elementary Linear Algebra, Chapter 2*: *Matrices*, pp 23-54, 2013.
7. W W L Chen, *Linear Algebra, Chapter 2: Matrices*, 2008.
8. Carl D. Meyer, *Matrix Analysis and Applied Linear Algebra, Chapter 3*: *Matrix Algebra*, pp 79-158, 2004.
9. E. H. Connell, *Elements of Abstract And Linear Algebra, Chapter 4: Matrices and Matrix Rings,* pp 53-64, 2004.
10. David C. Lay, *Linear Algebra And Its Applications, Chapter 2*: *Matrix Algebra*, pp 91-162, 4th Edition.
11. Robert A. Beezer, *A First Course in Linear Algebra, Chapter M*: *Matrices*, pp 197-302, Version 3.40.